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Phase shift analysis of the Landau–Lifshitz equation

Michael Svendsen† and Hans C Fogedby
Institute of Physics and Astronomy, University of Aarhus, DK-8000 Aarhus C, Denmark

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Abstract. We derive the complete spectrum of the Landau–Lifshitz equation using the Hirota method. Subsequently, we perform a phase shift analysis of spin-wave–kink and spin-wave–breather collisions, respectively. Finally, we use this result in order to derive the spin wave density of states in the presence of an arbitrary number of solitons.

1. Introduction

During the past decade, the Landau–Lifshitz (L–L) equation has attracted considerable attention. It was originally derived as a model for magnetic crystals [1], but has later been identified as a soliton-bearing system. Since the initial work by Sklyanin [2], who derived the compatibility condition and solved the initial-value problem using the inverse scattering transform (IST), the toroidal topology of the spectral problem has presented some difficulty [3–5]. The phase shift has been calculated explicitly by Bikbaev [6], who used a dressing procedure.

Papers based on the IST have dominated and only little attention has been paid to Hirota's approach [7], despite the fact that it offers a much more intuitive framework of handling nonlinearity. In this paper we draw on the solution by Bogdan and Kovalev [8], who solved the L–L equation in the pure N-soliton case by means of Hirota's bilinear operators. For a more thorough derivation, see also [9].

The statistical mechanics of a system is contained in the partition function, which is basically either a summation over all states or a summation over all energies weighted by the density of states. This last 'configurational approach' [10] is appealing since integrable models can be mapped onto ideal gas phenomenologies. In the present paper we shall not embark on a complete statistical mechanical analysis of the L–L equation, but perform a phase shift analysis yielding the complete density of states—a necessary prerequisite for a full treatment of the statistical problem.

Firstly, we derive the elementary solutions of the L–L equation by ordinary methods, and then derive the complete spectrum of solutions by means of Hirota's bilinear operators. Then, following the work of [11–13] on the sine–Gordon equation and the classical Heisenberg-chain, we calculate the phase shift of a spin wave colliding with an arbitrary number of solitons. Finally, we demonstrate how to derive the density of states of spin waves and discuss the result in some detail.

† Present address: Telecommunications Research Laboratory, Lyngsø Allé 2, DK-2970 Hørsholm, Denmark.

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2. The model

Basic solid state physics tells us [14] that magnetic ordering can be described by the Heisenberg model, $H = -J\sum_{ij} S_i \cdot S_j$, where the i,j index represents evenly spaced neighbouring spins S on some lattice. Improving this model by adding spatially inhomogenous terms, we can write

$$H = -J\sum_i S_i \cdot S_{i+1} + \sum_i \left(\alpha S_{1i}^2 + \beta S_{2i}^2\right) \tag{1}$$

where we have introduced anisotropy parameters α and β, and S_{1i}, S_{2i} are the x- and y-components of the spin field, respectively. We have furthermore limited the discussion to a spin chain and, without loss of generality, we have selected the z-axis as the axis of easiest magnetization.

In the limit of a large spin value we can treat the quantum mechanical spin operators as classical spins. Upon defining a spin field $S(x,t)$ and if we furthermore limit the discussion to long-wavelength excitations, an expansion of (1) to lowest order in the interspin distance a yields the Hamiltonian

$$H = \frac{1}{2}\int \left(\left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2 + (j_3 - j_1)\,S_1^2 + (j_3 - j_2)\,S_2^2\right)\mathrm{d}x \tag{2}$$

in some appropriate units and

$$j_3 > j_2 > j_1\,.$$

Here the spin field has been normalized to unity, the infinite ground-state energy has been subtracted, and the anisotropy parameters $j_1 \ldots j_3$ have been introduced in accordance with common notation.

Furthermore, the commutator algebra of the spin operators must be replaced by the Poisson-bracket algebra for classical spins in order to preserve the identity of the spin field as an angular momentum:

$$\{S_i(x,t), S_j(y,t)\} = -\delta(x-y)\sum_k \varepsilon^{ijk}\, S_k(x,t) \tag{3}$$

where $S(x,t)$ is defined by an underlying canonical basis. Since the Hamiltonian is the generator of time translations the equation of motion can be derived from $\mathrm{d}S/\mathrm{d}t = \{H(x,t), S(x,t)\}$, yielding

$$\frac{\mathrm{d}S}{\mathrm{d}t} = S \times \frac{\mathrm{d}^2 S}{\mathrm{d}x^2} + S \times JS \qquad J = \left\{\begin{array}{ccc} j_1 & 0 & 0 \\ 0 & j_2 & 0 \\ 0 & 0 & j_3 \end{array}\right\} \tag{4}$$

commonly referred to as the *Landau–Lifshitz equation*.

3. Elementary solutions

The first problem is of course to construct a canonical basis in which to define the spin field and which satisfies (3). This is done by defining the canonical coordinate $q(x,t)$ with the conjugate momentum $p(x,t)$ by

$$\begin{aligned} S_1 &= \sqrt{1-p^2}\cos q \\ S_2 &= \sqrt{1-p^2}\sin q \\ S_3 &= p\,. \end{aligned} \tag{5}$$

By insertion of (5) in (2) the Hamiltonian takes the form

$$\begin{aligned} H = \frac{1}{2}\int\bigg(\frac{1}{1-p^2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^2 + (1-p^2)\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2 \\ + (1-p^2)[(j_3-j_1)\cos^2 q + (j_3-j_2)\sin^2 q]\bigg)\,\mathrm{d}x \end{aligned} \tag{6}$$

and the equations of motion for q and p become

$$\begin{aligned} \dot{q} = \{H,q\} = -\frac{1}{1-p^2}\frac{\mathrm{d}^2p}{\mathrm{d}x^2} - \frac{p}{(1-p^2)^2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^2 \\ - p\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2 - p[(j_3-j_1)\cos^2 q + (j_3-j_2)\sin^2 q] \end{aligned} \tag{7}$$

$$\dot{p} = \{H,p\} = (1-p^2)\frac{\mathrm{d}^2q}{\mathrm{d}x^2} - 2p\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}q}{\mathrm{d}x} + \tfrac{1}{2}(j_2-j_1)(1-p^2)\sin 2q\,. \tag{8}$$

Since the Hamiltonian cannot be separated into a kinetic and a potential energy part, the usual particle-like interpretation of the model is not possible. Therefore, to construct the momentum Π one has to adopt another method. In view of the more general interpretation of the momentum as generator of space-translations, such an operator must be characterized by the property $\{\Pi, f\} = -\mathrm{d}f/\mathrm{d}x$, where f is an arbitrary function on phase-space.

The momentum is not uniquely defined, but as discussed in [2], $\Pi = \int q(\mathrm{d}p/\mathrm{d}x)\,\mathrm{d}x$ is to be preferred. The Poisson bracket of Π with H

$$\{\Pi, H\} = \frac{1}{2}\left[-\left(\frac{\mathrm{d}S}{\mathrm{d}x}\right)^2 + (j_3-j_1)\,S_1^2 + (j_3-j_2)\,S_2^2\right]_{\text{lower limit}}^{\text{upper limit}} \tag{9}$$

vanishes for both symmetric and anti-symmetric boundary conditions, assuming $\mathrm{d}S/\mathrm{d}x \to 0$ as $|x| \to \infty$. This proves H to be translationally invariant—or by reversing the order of Π and H: $\{H,\Pi\} = \mathrm{d}\Pi/\mathrm{d}t = 0$; that is, Π is a constant of motion since the Hamiltonian is the generator of time translations.

Due to lack of rotational symmetries in spin space none of the components of the total angular momentum are conserved. However, notice that the ground state is two-fold degenerate ($S_3 = \pm 1$).

3.1. Kinks

The simplest solution of (7) and (8) is obtained by assuming q constant: $q = q_0$. Insertion yields $\dot{p} = \frac{1}{2}(j_2 - j_1)(1 - p^2)\sin 2q_0$ which is readily integrated: $p(x,t) = \tanh(\omega t + f(x))$ where $\omega = \frac{1}{2}(j_2 - j_1)\sin 2q_0$ and f is an arbitrary function of x only. If we furthermore assume that f is a linear function, $f(x) = kx + \phi_0$, we obtain the soliton or *kink* solution

$$q = q_0 \qquad p = \tanh(kx + \omega t + \phi_0)\,. \tag{10}$$

The associated dispersion relation is obtained by demanding $\dot{q} = 0$ in (7):

$$\omega = \tfrac{1}{2}(j_2 - j_1)\sin 2q_0 \qquad k^2 = [(j_3 - j_1)\cos^2 q_0 + (j_3 - j_2)\sin^2 q_0]\,. \tag{11}$$

The motion of the kink is rather restricted in (ω, k)-space, with $\omega_{\max} = (j_2 - j_1)/2$ and $(j_3 - j_2) \leqslant k^2 \leqslant (j_3 - j_1)$. The phase velocity $v = \omega/k$ attains its maximum value at approximately $q_0 = \pi/4$ or $v_{\max} \approx \frac{1}{2}(j_2 - j_1)[(j_3 - j_1) + (j_3 - j_2)]^{-1/2}$. The conserved quantities are $E = 2|k|$ and $\Pi = 2q_0 \operatorname{sign} k$, respectively. In terms of the spin field the kink solution is

$$\begin{aligned} S_1 &= \cos q_0/\cosh(kx + \omega t + \phi_0) \\ S_2 &= \sin q_0/\cosh(kx + \omega t + \phi_0) \\ S_3 &= \tanh(kx + \omega t + \phi_0) \end{aligned} \tag{12}$$

which can be interpreted as follows: the magnetization is almost constant $|S_3| = 1$ along the chain except for a very narrow region of width $\sim 2a/|k|$ (remember the unit of length is the lattice distance a) which has been estimated in [1] to some 10^2 μm in real ferromagnets. Within this region the spin field rotates smoothly an angle π in a plane defined by e_3 and $(\cos q_0 e_1 + \sin q_0 e_2)$, where $e_1 \ldots e_3$ are the unit vectors of the coordinate system. This kind of solution is also known as a domain wall separating large regions of opposite spin.

3.2. Spin waves

The explicit form of the L–L equation is

$$\begin{pmatrix} \partial_t S_1 \\ \partial_t S_2 \\ \partial_t S_3 \end{pmatrix} = \begin{pmatrix} S_2\partial_x^2 S_3 - S_3\partial_x^2 S_2 + (j_3 - j_2)S_2S_3 \\ S_3\partial_x^2 S_1 - S_1\partial_x^2 S_3 + (j_1 - j_3)S_1S_3 \\ S_1\partial_x^2 S_2 - S_2\partial_x^2 S_1 + (j_2 - j_1)S_1S_2 \end{pmatrix} \tag{13}$$

where the abbreviated notation $\partial_x = \partial/\partial x$ has been introduced. Weak excitations may be characterized by $S_1 \approx 0$, $S_2 \approx 0$ and $S_3 \approx 1$, in which case (13) reduces to

$$\partial_t S_1 = -\partial_x^2 S_2 + (j_3 - j_2)S_2 \qquad \partial_t S_2 = \partial_x^2 S_1 + (j_1 - j_3)S_1 \tag{14}$$

—a linear equation. Some elementary manipulations will convince the reader that these equations are solved by the spin-wave solution

$$\begin{aligned} S_1 &\approx n\cos(kx + \omega t)\cdot\delta \\ S_2 &\approx -\sin(kx + \omega t)\cdot\delta \qquad \delta \ll 1 \\ S_3 &\approx 1 \end{aligned} \tag{15}$$

where

$$n^2 = \frac{k^2 + (j_3 - j_2)}{k^2 + (j_3 - j_1)} < 1 \tag{16}$$

and with dispersion relation

$$\omega^2 = [k^2 + (j_3 - j_2)][k^2 + (j_3 - j_1)]\,. \tag{17}$$

This is the kind of long-wavelength solutions always present in a many-body system with conserved quantities [15]. The individual spins precess around the e_3-axis in an elliptic movement with eccentricity $\epsilon = \sqrt{1-n^2}$, and since we have defined the anisotropy parameter of the e_1-axis greater than the corresponding parameter of the e_2-axis, the major axis of the ellipse is along the latter one.

4. Hirota's method

The starting point of the Hirota method is the definition of new differential operators D_x and D_t:

$$\mathrm{D}_t^m \mathrm{D}_x^n (a \cdot b) = (\partial_t - \partial_{t'})^m (\partial_x - \partial_{x'})^n a(x,t) b(x',t')|_{\substack{x'=x \\ t'=t}} \tag{18}$$

for non-negative integers m and n. Now, following [8] we define the initial transformation

$$S_1 + \mathrm{i}S_2 = \frac{2f^* g}{f^* f + g^* g} \qquad S_3 = \frac{f^* f - g^* g}{f^* f + g^* g} \tag{19}$$

where f and g are complex functions, which, when inserted in the equations of motion (4), can be reduced to the tri-linear representation

$$\begin{aligned} &f^*\left[(\mathrm{iD}_t + \mathrm{D}_x^2 - a(1+b))(g \cdot f)\right] + g^*\left[\mathrm{D}_x^2(g \cdot g) - a(1-b)(f \cdot f)\right] = 0 \\ &g^*\left[(-\mathrm{iD}_t + \mathrm{D}_x^2 - a(1+b))(g \cdot f)\right] + f^*\left[\mathrm{D}_x^2(f \cdot f) - a(1-b)(g \cdot g)\right] = 0 \end{aligned} \tag{20}$$

where

$$a = \frac{j_3 - j_1}{2} \qquad b = \frac{j_3 - j_2}{j_3 - j_1}\,. \tag{21}$$

Later, Hirota [16] demonstrated the following *bilinear* representation of the L–L equation

$$\begin{aligned} &\mathrm{D}_x(f^* \cdot f + g^* \cdot g) = 0 \\ &(\mathrm{iD}_t + \mathrm{D}_x^2)(f \cdot g^*) - a(fg^* + f^* g) - ab(fg^* - f^* g) = 0 \\ &(\mathrm{iD}_t - \mathrm{D}_x^2)(f^* \cdot f - g^* \cdot g) = 0 \end{aligned} \tag{22}$$

which will serve as the starting point for our investigation. If we interpret products like fg^* as $f \cdot g^*$ and the bilinear differential operators as ordinary differential operators acting on the 'functions' $f \cdot g^*$, equations (22) certainly look linear. In what sense this is actually true, we shall explore below.

4.1. One-soliton solution

The general idea when solving the bilinear equations [17] is to assume a series expansion of functions f and g in some arbitrary parameter ε,

$$f(x,t)=\sum_{n=0}^{\infty}\varepsilon^n f_n(x,t) \qquad g(x,t)=\sum_{n=0}^{\infty}\varepsilon^n g_n(x,t) \tag{23}$$

where there is no reason to worry about convergence, since we shall soon prove that both series are terminated after a finite number of terms. Furthermore, since there is no restrictions on the choice of the expansion parameter ε, we shall choose $\varepsilon = 1$ whenever convenient.

Now, since ε is an arbitrary parameter, equations (22) *have* to be satisfied for each power of ε when inserting the expansions. To zeroth order in ε we find upon insertion of (23) in (22):

$$\begin{aligned}
&\mathrm{D}_x(f_0^*\cdot f_0+g_0^*\cdot g_0)=0\\
&(\mathrm{iD}_t+\mathrm{D}_x^2)(f_0\cdot g_0^*)-a(f_0g_0^*+f_0^*g_0)-ab(f_0g_0^*-f_0^*g_0)=0\\
&(\mathrm{iD}_t-\mathrm{D}_x^2)(f_0^*\cdot f_0-g_0^*\cdot g_0)=0\,.
\end{aligned}\tag{24}$$

This equation is solved if we choose $f_0 = 1$ and $g_0 = 0$.

To first order in ε we obtain the equations

$$\begin{aligned}
&\mathrm{D}_x(1\cdot f_1+f_1^*\cdot 1)=0\\
&(\mathrm{iD}_t+\mathrm{D}_x^2)(1\cdot g_1^*)-a(g_1^*+g_1)-ab(g_1^*-g_1)=0\\
&(\mathrm{iD}_t-\mathrm{D}_x^2)(1\cdot f_1+f_1^*\cdot 1)=0
\end{aligned}\tag{25}$$

which are satisfied by $f_1 = 0$ and $g_1 = \mathrm{e}^{\eta}$, where $\eta = kx+\omega t+\eta_0$ and $\eta_0 = \eta_0'+\mathrm{i}\eta_0''$ is some (complex) phase. The dispersion relation is

$$\omega=-a(1-b)\sin 2\eta_0'' \qquad k^2=2ab+2a(1-b)\cos^2\eta_0'' \tag{26}$$

—identical to the one already derived in (11).

Finally, to second order in ε, f_2 and g_2 have to satisfy

$$\begin{aligned}
&\mathrm{D}_x(1\cdot f_2+f_2^*\cdot 1+g_1^*\cdot g_1)=0\\
&(\mathrm{iD}_t+\mathrm{D}_x^2)(f_1\cdot g_1^*+1\cdot g_2^*)-a[(g_2^*+f_1g_1^*)+(g_2+f_1^*g_1)]\\
&\qquad\qquad -ab[(g_2^*+f_1g_1^*)-(g_2+f_1^*g_1)]=0\\
&(\mathrm{iD}_t-\mathrm{D}_x^2)(1\cdot f_2+f_2^*\cdot 1-g_1^*\cdot g_1)=0
\end{aligned}\tag{27}$$

which, along with all higher-order equations, are solved by $f_n = 0$ and $g_n = 0$ for $n \geqslant 2$. As mentioned earlier, the series are terminated after a finite number of terms. Collecting all terms we conclude that there exists a solution $f = 1$ and $g = \mathrm{e}^{\eta}$ which, when inserted in (19), yields the kink already derived in (12).

4.2. Two-soliton solution

Firstly, following the same line of arguments as in section 4.1 we can again set $f_0 = 1$ and $g_0 = 0$. Furthermore, since the first-order equations (25) are *linear* in g_1 due to the properties of D_t and D_x, we can add any number of terms e^{η_i} to g_1. Then equations (25) simply separate into independent sets of terms which are all identically zero provided ω_i and k_i satisfies the dispersion relation. It is in this sense that the bilinear representation linearizes the L–L equation, thereby providing a nonlinear superposition principle for the evolution equation.

To demonstrate this point in the case of the two-soliton solution, we put

$$f_1 = 0 \qquad g_1 = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} \tag{28}$$

which according to the above arguments is a solution of (25). Unfortunately, this is not the end of our discussion, and we must also consider modifications of the higher-order terms in the series expansion of f and g. Unlike the kink solution, terms like $\mathrm{D}_x(g_1^* \cdot g_1)$ now become non-vanishing in (27):

$$\mathrm{D}_x(g_1 \cdot g_1^*) = \mathrm{D}_x(\mathrm{e}^{\eta_1^*} \cdot \mathrm{e}^{\eta_2}) + \mathrm{D}_x(\mathrm{e}^{\eta_2^*} \cdot \mathrm{e}^{\eta_1}) \neq 0 \tag{29}$$

and the second-order equations are *not* solved by $f_2, g_2 = 0$. Following the standard procedure, we then assume

$$f_2 = \mathrm{e}^{\eta_1+\eta_2+\theta_{12}} \qquad g_2 = 0 \tag{30}$$

where the *phase shift* θ_{ij} has been introduced. Inserting in (27) and after some manipulations, we obtain the relation

$$\mathrm{e}^{\theta_{ij}} = \frac{k_j - k_i}{k_j + k_i}\,\frac{\omega_i k_j^2 - \omega_j k_i^2 - 2ab(\omega_i - \omega_j)}{\omega_i k_j^2 + \omega_j k_i^2 - 2ab(\omega_i + \omega_j)}\,. \tag{31}$$

This solves the second-order equations, but to be sure that we can terminate the series after the second-order terms, we have to check the third-order equations too. An elementary but laborious calculation shows that the third-order equations are satisfied if the individual solitons satisfy the dispersion relation in (26).

Summarizing the two soliton solution, we find

$$f = 1 + \mathrm{e}^{\eta_1+\eta_2+\theta} \qquad g = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}\,. \tag{32}$$

In a reference system where the centre of mass of the two solitons is at rest

$$-\eta_{01}'' = \eta_{02}'' = \gamma \qquad k_1 = k_2 = k \qquad \omega_1 = -\omega_2 = \omega \tag{33}$$

the two-soliton solution in terms of spin variables becomes

$$\begin{aligned} S_1(x,t) &= \frac{2\mathrm{e}^{\theta/2}\cos\gamma\cosh kx\cosh\omega t}{\mathrm{e}^{\theta}\cosh^2 kx + \sinh^2\omega t + \cos^2\gamma} \\ S_2(x,t) &= \frac{-2\mathrm{e}^{\theta/2}\sin\gamma\cosh kx\sinh\omega t}{\mathrm{e}^{\theta}\cosh^2 kx + \sinh^2\omega t + \cos^2\gamma} \\ S_3(x,t) &= \frac{\mathrm{e}^{\theta}\cosh^2 kx - \sinh^2\omega t - \cos^2\gamma}{\mathrm{e}^{\theta}\cosh^2 kx + \sinh^2\omega t + \cos^2\gamma}\,. \end{aligned} \tag{34}$$

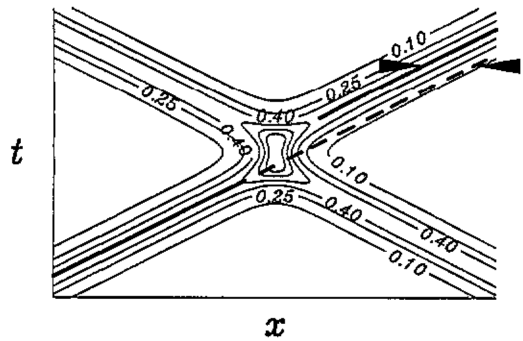

Figure 1. Contour map of the S_1 component of the two-soliton solution (au). The full lines trace the CM of one of the solitons and the broken line indicates the path of that CM in the absence of collisions. Long after the interaction the only reminiscence of the interaction is a phase shift indicated by the two arrows.

The phase shift is the undisputed trademark of a soliton. In figure 1 a contour map of the S_1-component of the two-soliton solution in (34) is shown along with lines tracing the path of the centre of mass (CM) of one of the solitons. The broken line indicates the path of the CM had the solitons not interacted. The CM traces different lines before and after the collision, indicating that long before and long after the collision the two solitons move independently of each other with constant velocity. Within a narrow region of the point of contact the two solitons interact nonlinearly before they break contact and resume their original form. The only reminiscence of the collision is a characteristic phase shift θ, derived above, indicated by the two arrows.

4.3. N-soliton solution

Clearly, the above procedure can be extended to N terms in g_1, and following the procedure outlined above, the corresponding solutions can be constructed with rapidly increasing labour. Based on the tri-linear representation in (20), Bogdan and Kovalev [8] have suggested the general solution

$$g = \sum_{m=0}^{[(N-1)/2]} \sum_{N^C_{2m+1}} \mathrm{e}^{\eta_{i_1}+\cdots+\eta_{i_{2m+1}}+\theta(i_1,\ldots,i_{2m+1})}$$
$$f = \sum_{m=0}^{[N/2]} \sum_{N^C_{2m}} \mathrm{e}^{\eta_{i_1}+\cdots+\eta_{i_{2m}}+\theta(i_1,\ldots,i_{2m})} \tag{35}$$

where the total phase shift $\theta(i_1,\ldots,i_n)$ is given by

$$\theta(i_1,\ldots,i_n) = \begin{cases} \sum_{k<l} \theta_{i_k i_l} & n \geqslant 2 \\ 0 & n = 0,1 \end{cases} \tag{36}$$

where $\theta_{i_k i_l}$ are the individual phase shifts in (31), $[N/2]$ is the lowest integer greater than $N/2$ and N^C_n represents summation over all combinations of n elements in N. Notice, that the additivity of the total phase shift is just another manifestation of the nonlinear superposition principle.

Is there an upper limit to the number of solitons, N? The question of completeness has been proved in some cases—see e.g. [18] for a proof in the case of the Korteweg–de Vries equation—but to the knowledge of the authors such a proof has not been published for the L–L equation. Such proofs are usually by mathematical induction and are very tedious. We shall not attempt such an approach and instead rely on the corresponding proof of the IST.

In this paper we use Hirota's Method as a convenient way of constructing *explicit* solutions, as opposed to the IST which offers a more formal framework of handling nonlinearity. The spectrum derived by us exhibits exactly the same phenomenology as the (complete) spectrum derived by the IST [2,9] and is in this sense complete.

4.4. *Breathers and spin waves*

Breathers are special kinds of solitons containing an internal degree of freedom. They can be interpreted as a bound state of a number of kinks and can be obtained by performing an analytical continuation of ω_i and k_i in (26) [11]. The simplest case is the analytical continuation of the two soliton solution. For convenience we use the parametrization

$$\begin{aligned} &\eta''_{01} = \eta''^{*}_{02} = \gamma^* && \gamma = u + \mathrm{i}v \\ &\omega_1 = \omega_2^* = \omega && \omega = -a(1-b)\sin(2\gamma) \\ &k_1 = k_2^* = k && k^2 = 2a\left[b + (1-b)\cos^2\gamma\right] \end{aligned} \tag{37}$$

where u and v are real. A brief calculation yields

$$\begin{aligned} S_1 &= 4\mathrm{e}^{\theta/2}\frac{\cos u\,\cosh v\,\cosh\Delta'\,\cos\Delta'' - \sin u\,\sinh v\,\sinh\Delta'\,\sin\Delta''}{\mathrm{e}^\theta\cosh 2\Delta' + \mathrm{e}^\theta\cos 2u + \cos 2\Delta'' + \cosh 2v} \\ S_2 &= -4\mathrm{e}^{\theta/2}\frac{\sin u\,\cosh v\,\sinh\Delta'\,\cos\Delta'' + \cos u\,\sinh v\,\cosh\Delta'\,\sin\Delta''}{\mathrm{e}^\theta\cosh 2\Delta' + \mathrm{e}^\theta\cos 2u + \cos 2\Delta'' + \cosh 2v} \\ S_3 &= \frac{\mathrm{e}^\theta\cosh 2\Delta' + \mathrm{e}^\theta\cos 2u - \cos 2\Delta'' - \cosh 2v}{\mathrm{e}^\theta\cosh 2\Delta' + \mathrm{e}^\theta\cos 2u + \cos 2\Delta'' + \cosh 2v} \end{aligned} \tag{38}$$

where we have defined the phases $\Delta' = k'x + \omega' t + \theta/2$ and $\Delta'' = k''x + \omega'' t$. The internal degree of freedom is explicit in the oscillatory nature of the harmonic functions. The phase shift turns out to be real:

$$\mathrm{e}^\theta = \frac{k''}{k'}\frac{\mathrm{Im}\left(\omega k^{*2} - 2ab\omega\right)}{\mathrm{Re}\left(\omega k^{*2} - 2ab\omega\right)} \tag{39}$$

that is, the phases of the harmonic functions are unaffected by the 'self-interaction'.

The oscillatory nature of the breathers suggests that spin waves can be derived in some appropriate limit. Since spin waves are extended states, we start by expanding about $k' = \omega' = 0$ where breathers become non-localized. Now, defining parameters ϵ and μ as

$$u = \frac{\pi}{2} + \epsilon \qquad v = v_0 + \mu \tag{40}$$

where v_0 is determined by the implicit expression, $\sinh v_0 = \sqrt{b/(1-b)}$, we can expand ω to second order in ϵ in order to obtain

$$\omega(\epsilon) = -\mathrm{i}A_\omega + B_\omega\epsilon + \mathrm{i}C_\omega\epsilon^2 \tag{41}$$

where

$$\begin{aligned}
A_\omega &= a\left[2\sqrt{b}\cosh 2\mu + (1+b)\sinh 2\mu\right] \\
B_\omega &= 2a\left[(1+b)\cosh 2\mu + 2\sqrt{b}\sinh 2\mu\right] \\
C_\omega &= 2A_\omega
\end{aligned}$$

and similarly for k:

$$k(\epsilon) = \mathrm{i}A_k - B_k\epsilon + \mathrm{i}C_k\epsilon^2 \tag{42}$$

where

$$\begin{aligned}
A_k &= \sqrt{2a\left[(2b+1)\sinh^2\mu + \sqrt{b}\sinh 2\mu\right]} \\
B_k &= 2a\left[\sqrt{b}\cosh 2\mu + (1+b)\sinh 2\mu/2\right] A_k^{-1} \\
C_k &= \tfrac{1}{2}\left[B_k^2 - 2a\left((1+b)\cosh 2\mu + 2\sqrt{b}\sinh 2\mu\right)\right] A_k^{-1}.
\end{aligned}$$

Clearly, the limit $\epsilon \to 0$ now corresponds to $k', \omega' \to 0$. The critical factor turns out to be the phase shift which in this limit diverges as

$$\mathrm{e}^\theta \sim -F(\mu)\epsilon^{-2} \qquad \epsilon \ll 1 \tag{43}$$

where

$$F(\mu) = \frac{A_k\,A_\omega(A_k^2 + 2ab)}{B_k(B_\omega(2ab + A_k^2) - 2A_k\,B_k\,A_\omega)}. \tag{44}$$

Insertion of this result along with the expansions (41) and (42) in (38) yields, to leading order in ϵ:

$$\begin{aligned}
S_1 &= \tanh(v_0+\mu)\sin(k''x + \omega''t)\cdot\epsilon^2 + \mathcal{O}(\epsilon^3) \\
S_2 &= -\cos(k''x + \omega''t)\cdot\epsilon^2 + \mathcal{O}(\epsilon^3) \\
S_3 &= 1 + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{45}$$

which shows that a breather collapses into a spin wave in the limit $\epsilon \to 0$. This result should be compared to (15); it is easy to check that these solutions are identical.

This concludes the discussion of Hirota's method, since we have now derived all the elementary excitations: kinks, breathers and spin waves. Furthermore, by choosing different combinations of real and complex ωs and ks, we can use the nonlinear superposition principle to construct the complete solution comprising any combination of these excitations thereby providing a reasonably handy method of handling the full spectrum of solutions of the intrinsically nonlinear problem.

5. Phase shift analysis

Following [11, 13] we assume that the ferromagnet extends a distance L along the x-axis, where the limit $L \to \infty$ is consistent with the use of Hirota's method. In the absence of solitons, the only excitations present are spin waves parametrized by the wave vector k. If we furthermore impose periodic (anti-periodic) boundary conditions, these continuum states are allowed *only* if they satisfy

$$Lk = n2\pi + \kappa\pi \qquad \pm 1, \ldots \tag{46}$$

where $\kappa = 0, 1$ corresponds to periodic and anti-periodic boundary conditions, respectively. The density of states is unaffected by this artifice, and in the absence of solitons we shall denote it $\rho_0(k) = \partial_k n = L/2\pi$.

During collisions with solitons the spin waves suffer a phase shift δ, which alters the above condition to

$$Lk = n2\pi + \delta(k) + \kappa\pi \tag{47}$$

where $\delta(k)$ is the *explicit* phase shift of the spin wave due to collision with an arbitrary number of solitons. This yields the simple expression for the density of states of the continuum modes in the presence of solitons:

$$\rho(k) = \rho_0(k) + \frac{\partial_k \delta(k)}{2\pi} \tag{48}$$

valid in the limit $L \to \infty$.

5.1. Collisions with kinks

As explained in section 4, Hirota's method offers a systematic way of treating the spectrum of solutions. Since we know in what limit a breather collapses into a spin wave, we can avoid the problem of the diverging phase shift by first calculating the phase shift of a breather due to collision with a kink, and then derive the corresponding result for a spin wave by performing the limit discussed above.

We start by constructing the combined system of a breather and a kink. We parametrize the breather as in (37) and to avoid confusion of the individual solitons, we shall parametrize the kink in upper case letters, i.e. $\omega_3 = \Omega$ and $k_3 = K$.

Treating the breather as one excitation we can calculate the phase shift by inserting (the complex) k, ω and (the real) K, Ω in (31). In particular, when collapsing the breather into a spin wave by taking the limits $k', \omega' \to 0$, we arrive at

$$e^{\theta} = e^{i\delta^{(k)}}$$

where superscript (k) refers to collision with a kink and

$$\delta^{(k)} = \pi + 2\arctan\left(\frac{K\omega(K^2 - 2ab) - k\Omega(k^2 + 2ab)}{k\omega(K^2 - 2ab) + K\Omega(k^2 + 2ab)}\right). \tag{49}$$

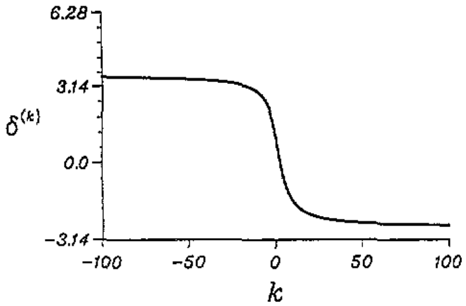

Figure 2. Phase shift $\delta^{(k)}$ of spin wave due to collision with a kink as a function of wave vector k. Parameters are $a = 10$, $b = 5$ and $K = 5.0$.

Here the spin wave has been re-parametrized by $k'' \to k$ and $\omega'' \to \omega$. A plot of the phase shift as a function of k is shown in figure 2.

The imaginary part of the total phase shift is associated with the phases of the harmonic functions, and is therefore the phase shift of the spin wave. This can also be verifed directly by writing the solution in (19) explicitly in terms of the real and imaginary parts of the complex η_{0i}s, k_is, ω_is and a general (complex) phase shift.

As discussed previously, the phase shift of a soliton colliding with several solitons is an additive quantity. The phase shift due to collision with N kinks is then simply the sum of the individual contributions. Enumerating the individual kinks we obtain, defining $\delta_i^{(k)}$ according to (49),

$$\delta = \sum_i \delta_i^{(k)} . \tag{50}$$

This sum completely determines the phase shift of a spin wave by collision with an arbitrary number of kinks.

5.2. Collisions with breathers

To calculate the influence of breathers on the density of spin wave states, we define the combined system of two breathers:

$$\begin{aligned} &\text{breather \#1} \begin{cases} \omega_1 = \omega_2^* = \omega \\ k_1 = k_2^* = k \end{cases} \\ &\text{breather \#2} \begin{cases} \omega_3 = \omega_4^* = \omega_b \\ k_3 = k_4^* = k_b . \end{cases} \end{aligned} \tag{51}$$

The phase shift of #1 can be calculated by insertion of (51) into (31). Finally, collapsing #1 into a spin wave, we arrive at the (imaginary part of the) phase shift:

$$\begin{aligned} \delta^{(b)} = \arctan\left(\frac{k_b'' - k}{k_b'}\right) - \arctan\left(\frac{k_b'' + k}{k_b'}\right) \\ + \arctan\left(\frac{\omega(k_b'^2 - k_b''^2) + \omega_b'' k^2 - 2ab(\omega - \omega_b'')}{\omega_b'(k^2+1) - 2\omega k_b' k_b''}\right) \\ + \arctan\left(\frac{\omega(k_b'^2 + k_b''^2) - \omega_b'' k^2 - 2ab(\omega + \omega_b'')}{\omega_b'(k^2+1) + 2\omega k_b' k_b''}\right) \end{aligned} \tag{52}$$

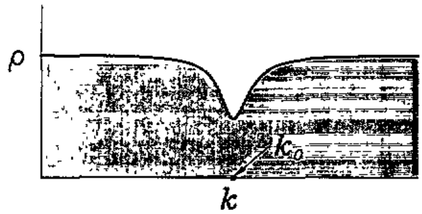

Figure 3. Density of spin wave states in the presence of one soliton (au). The dip is caused by the soliton which 'traps' exactly one spin wave mode k_0.

where again the spin wave has been re-parametrized by $k'' \to k$ and $\omega'' \to \omega$.

Equation (52) solves the density of states of spin waves in the presence of a breather. The total effect of M breathers adds up, $\delta = \sum_i \delta_i^{(b)}$, exactly as in the case of N kinks. Notice that in the limit where breather #2 collapses into another spin wave, that is, in the limit $k_b', \omega_b' \to 0$, the above phase shift vanishes, consistent with the fact that spin waves satisfy the *linear* superposition principle.

5.3. Discussion

The effect of the presence of a kink on the density of states of spin waves is depicted in figure 3. The kink causes a 'dip' in the density of states at some characteristic wavevector k_0, which has to be determined from the implicit relation $\partial_k^2 \delta^{(k)}\big|_{k_0} = 0$. The 'dip' corresponds to the mode k_0 being 'trapped' by the kink since k_0 depends on a, b as well as the kink-parameters K, Ω.

Such an interpretation is corroborated by the following calculation: the net change in the density of spin waves ρ is the difference after and before the collision, $\Delta\rho = \partial_k \delta^{(k)}/2\pi$. The net change in the number of spin modes available is the integral of this difference:

$$\Delta N = \int_{-\infty}^{\infty} \Delta\rho \, \mathrm{d}k = \frac{1}{2\pi}\left[\delta^{(k)}(\infty) - \delta^{(k)}(-\infty)\right] = -1 \tag{53}$$

which is evident from figure 2. Basically this is just *Levinson's theorem*, known from scattering theory, and demonstrates the intimate connection between soliton theory and scattering theory. Finally, we note that a similar calculation is true for the collision with breathers as well.

It should now be clear that a complete decomposition of the spectrum of solutions of the L–L equation is possible forming the basis of the 'configurational' approach [10]. The mapping onto an ideal gas phenomenology consisting of two kinds of 'particles', namely kinks and breathers, and 'radiation', i.e. spin waves, makes complicated integrations over phase space unnecessary when calculating statistical mechanical properties, since they can be replaced by summations over all excited fundamental modes (weighted by their proper density of states). Finally we note, that such a decomposition can also be done in the IST-formalism, where the equivalent action angle representation can be derived as well [2, 9].

References

[1] Landau L D and Lifshitz E M 1965 *Phys. Z. Sowjet* **8** 153 (Engl. transl.: On the theory of the dispersion of magnetic permeability in ferromagnetic bodies *Collected Papers of L D Landau* (New York: Pergamon Press))

[2] Sklyanin E K 1979 On the complete integrability of the Landau–Lifshitz equation *LOMI preprint* Leningrad
[3] Borisov A B 1983 The Hilbert problem for matrices and a new class of integrable equations *Lett. Math. Phys.* 7 195
[4] Rodin Y L 1983 The Riemann boundary problem on a torus and the inverse scattering problem for the Landau–Lifshitz equation *Lett. Math. Phys.* 7 3
[5] Borisov A B 1986 Direct and inverse scattering problem for the Landau–Lifshitz equation *Sov. Phys.–Dokl.* **31** 482
[6] Bikbaev R F 1989 Asymptotic behaviour as $t \to \infty$ of the solution to the Cauchy problem for the Landau–Lifshitz equation *Theor. Math. Phys.* **77** 1117
[7] Hirota R 1971 Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons *Phys. Rev. Lett.* **27** 1192
[8] Bogdan M M and Kovalev A S 1980 Exact multisoliton solution of one-dimensional Landau–Lifshitz equations for an anisotropic ferromagnet *JETP Lett.* **31** 425
[9] Svendsen M 1992 Solitons in the Landau–Lifshitz equation *Masters' thesis* University of Aarhus, Denmark
[10] Bishop A R and Schneider T 1979 *Solitons and Condensed Matter Physics* (Berlin: Springer)
[11] Currie J F 1977 Aspects of exact dynamics for general solutions of the sine–Gordon equation with applications to domain walls *Phys. Rev.* A **16** 1692
[12] Fogedby H C 1980 Solitons and magnons in the classical Heisenberg chain *J. Phys. A: Math. Gen.* **13** 1467
[13] Fogedby H C 1980 Magnon–soliton phase shift analysis of the classical continuous Heisenberg chain *Phys. Scr.* **22** 404
[14] Mattis D C 1981 *The Theory of Magnetism* vol 1 (Berlin: Springer)
[15] Foster D 1975 *Hydrodynamics, Fluctuations, Broken Symmetry and Correlation Functions* (New York: Benjamin)
[16] Hirota R 1982 Bilinearization of soliton equations *J. Phys. Soc. Japan* **51** 323
[17] Hirota R and Satsuma J 1976 A variety of nonlinear network equations generated from the Bäcklund transformation for the Toda lattice *Suppl. Prog. Theor. Phys.* **59** 64
[18] Matsuno Y 1984 *Bilinear Transformation Method* (London: Academic)